Analytic Central Orbits and their Transformation Group

D. Lynden-Bell* and S. Jin$\dagger$
Institute of Astronomy, University of Cambridge, Madingley Road, Cambridge CB3 0HA
Clare College, Trinity Lane, Cambridge CB2 1TL

ABSTRACT
A useful crude approximation for Abelian functions is developed and applied to orbits. The bound orbits in the power-law potentials $Ar^{-\alpha}$ take the simple form $(\ell/r)^k = 1 + e\cos(m\phi)$, where $k = 2 - \alpha > 0$ and ℓ and e are generalisations of the semi-latus-rectum and the eccentricity. m is given as a function of 'eccentricity'. For nearly circular orbits m is $\sqrt{k}$, while the above orbit becomes exact at the energy of escape where e is one and m is k. Orbits in the logarithmic potential that gives rise to a constant circular velocity are derived via the limit $\alpha \to 0$. For such orbits, r^2 vibrates almost harmonically whatever the 'eccentricity'. Unbound orbits in power-law potentials are given in an appendix. The transformation of orbits in one potential to give orbits in a different potential is used to determine orbits in potentials that are positive powers of r. These transformations are extended to form a group which associates orbits in sets of six potentials, e.g. there are corresponding orbits in the potentials proportional to $r, r^{-\frac{2}{3}}, r^{-3}, r^{-6}, r^{-\frac{4}{3}}$ and r^4. A degeneracy reduces this to three, which are r^{-1}, r^2, and r^{-4} for the Keplerian case. A generalisation of this group includes the isochrone with the Kepler set.

Key words: celestial mechanics — galaxies: kinematics and dynamics

1 INTRODUCTION

Since schooldays when we encountered the rigid pendulum, most of us have been frustrated by our inability to integrate in elementary terms Abelian expressions of the form $\int [S(u)]^{-\frac{1}{2}}\, du$, where $S(u)$ has simple zeros at u_a and $u_p \geqslant u_a$ but is not quadratic. In practice S usually depends linearly or quadratically on parameters which we shall call ε and h, and its zeros u_p and u_a depend on ε and h often in quite complicated ways. In Appendix A we relieve this frustration by showing how for each pair of u_a and u_p, S may be replaced at lowest order by a quadratic function **with the same zeros** and the integral may be approximately evaluated parametrically via perturbation theory.

We do not have to solve $S(\varepsilon, h, u)$ for its zeros u_p and u_a. Instead we regard u_p and u_a as parameters and then easily find the $\varepsilon(u_a, u_p)$ and $h(u_a, u_p)$ to which they correspond. The process of replacing S by a different quadratic function for each pair of zeros u_p and u_a we call **quadrating** (after the old verb 'to quadrate' which means 'to make square'). Surely using 'quadrate' to mean 'to make quadratic' is not too great an extension! As the simple, though crude, method developed can be applied to a far wider class of problems

* e-mail: dlb@ast.cam.ac.uk
$\dagger$ e-mail: shoko@ast.cam.ac.uk

than those encountered here, we have mentioned it first in the introduction.

Orbits of the general form

$$(\ell/r)^k = 1 + e\cos(m\phi) \qquad (1)$$

have a long history. Newton in Principia (1687) showed that orbits of this form with $k = 1$ occurred when the central force was an inverse square law supplemented by an inverse cube force. His celebrated theorem on revolving orbits demonstrates that if $r = r(\phi)$ is an orbit of angular momentum h under any central force $F(r)\hat{\mathbf{r}}$, then $r = r(m\phi)$ is an orbit of angular momentum mh under the central force $\left[F - (m^2 - 1) h^2 r^{-3}\right]\hat{\mathbf{r}}$. Newton also pointed out that $r(t)$ was the same for both orbits, so, if the new orbit were viewed from axes revolving at the rate $(m - 1)\dot{\phi}$, then the two orbits would have the same shape. But notice that $(m - 1)\dot{\phi}$ is not a constant rotation rate, but speeds up when r is small and slows down when r is large. When we view the orbit from axes that rotate uniformly at the same mean rate $(m - 1) < \dot{\phi} >$, the orbits can have very different shapes involving figures of eight for the more eccentric ones (Lynden-Bell & Lynden-Bell 1995).

Recently in a fine paper, Struck (2006) showed that orbits of moderate or low eccentricity in logarithmic or power-law potentials with or without cores were well approximated by analytic orbits of the form (1). His approximate orbits

are surprisingly accurate. Struck was, in part, stimulated to find this result by a paper by Touma & Tremaine (1997) that demonstrated the richness of the resonances in the perturbation theory of these systems. Valluri et al. (2005) have emphasised that the apsidal precession found in non-inverse-square orbits is significantly dependent on the eccentricity of the orbit involved.

Surprisingly, we have been led to orbits of the form (1) by looking at orbits of extreme eccentricity $e = 1$, where Struck's methods did not give accurate results. Standard works on orbits, Boccaletti & Pucacco (1996), Contopoulos (2002) and Binney & Tremaine (1987), do not point out that the orbits of zero energy in power-law potentials can be exactly solved analytically. The same variables can be used to solve the nearly circular orbits. Since both highly eccentric and small eccentricity orbits can be so solved, it would be surprising if there were **not** a good approximation, based on the same variables, that interpolated between $e = 1$ and $e = 0$. Struck's methods do this well for small and moderate eccentricities. Here we show that all orbits are well approximated by analytic orbits of the form (1) for $0 \leqslant e \leqslant 1$.

Here ℓ and e are generalisations of the semi-latus-rectum and the eccentricity. The potential is $Ar^{-\alpha} = Ar^{k-2}$; this parameterisation, using k rather than α, is chosen in this section to simplify equations (2) and (3) below. If r_a and r_p are the apocentric and pericentric distances the generalised eccentricity is given by

$$e = \frac{r_p^{-k} - r_a^{-k}}{r_p^{-k} + r_a^{-k}} = \frac{r_a^k - r_p^k}{r_a^k + r_p^k} \ , \tag{2}$$

and the generalised semi-latus-rectum is given by ℓ where

$$\ell^{-k} = \frac{1}{2}\left(r_p^{-k} + r_a^{-k}\right) \ . \tag{3}$$

We note that for the Kepler case $k = 1$ and the above formulae all reduce to the usual ones. We write $\ell = Lr_c$, where $r_c(h)$ is the radius of the circular orbit of angular momentum h. We have $r_c^k = (2-k)^{-1}h^2/A$. The dimensionless parameters L and m are functions of e. Orbits of small eccentricity have $m = \sqrt{k}$, while those with $e = 1$ have $m = k$. We find the orbits in the $-V^2 \ln r$ potential from the limiting case $k \to 2$.

In Appendix A we show how to improve the accuracy of our orbits via perturbation theory; however for most purposes the simplicity of the initial approximation (1) outweighs the extra complication that accompanies greater accuracy. Our methods can be applied to non-power law potentials (see Kalnajs (1979)) but here, for simplicity, we limit ourselves to power laws.

While our methods can be extended to unbound orbits, the results are less pleasing so they are consigned to Appendix B.

In section 3 we use the transformation theory of Newton, Bohlin, Arnold and others to transform our orbits for $0 < k < 2$ into orbits in potentials with positive powers of r. We show how that theory can be extended naturally to give a set of transformations that form a group. We develop the subgroup of switch transformations and show that orbits in the potentials $\psi \propto r^{k-2}$ are conjugate to orbits in the potentials $r^{2(2-k)/k}, r^{-k}, r^{2k/(2-k)}, r^{-4/k}$ and $r^{-4/(2-k)}$. These transformations are not restricted to power laws, al-

though special simplifications occur for them. Applications are made to Plummer's law.

It is shown that the full group has a transformation that connects the Keplerian potential to the isochrone.

2 ANALYTIC ORBITS

2.1 General Orbits in Potentials with $0 < k < 2$

Those looking for orbits in potentials with powers k outside the above range should consult section 3.

A general orbit of specific energy ε and specific angular momentum h in the power-law potential $\psi = Ar^{k-2}$ has, in the usual notation, $r^2\dot\phi = h$ and

$$\dot r = \sqrt{2\varepsilon + 2Ar^{k-2} - h^2r^{-2}} \ .$$

Now

$$d\phi = \dot\phi dt = hr^{-2}dr/\dot r = \frac{dr}{r\sqrt{2\varepsilon h^{-2}r^2 + 2Ah^{-2}r^k - 1}} \ . \tag{4}$$

In place of r we shall use a dimensionless variable which generalises the $u \left(= \frac{1}{r}\right)$, so useful in the Keplerian case:

$$u = h^2/(Ar^k) \ . \tag{5}$$

We also define a dimensionless energy

$$E = \left(\frac{\varepsilon}{A}\right)\left(\frac{h^2}{A}\right)^{(2-k)/k} \ . \tag{6}$$

Now $k \, dr/r = -du/u$ so we may rewrite (4) in terms of u:

$$k \, d\phi = -\left[S(u)\right]^{-\frac{1}{2}} du \ , \tag{7}$$

where

$$S(u) = 2Eu^\sigma + 2u - u^2 \ , \tag{8}$$

and $\sigma = 2(k-1)/k$, which is less than one for $0 < k < 2$. For the marginally bound orbits $\varepsilon = 0$, the u^σ term in (8) disappears so we may integrate (7) exactly. Substituting $u = 1 + \cos\eta$ reduces (7) to $k \, d\phi = d\eta$. Choosing the zero of ϕ at that pericentre where $\eta = 0$, we have $\eta = k\phi$, so the solution for the orbit is of the form of equation (1) with $e = 1$ and $m = k$:

$$u = (\ell/r)^k = 1 + \cos(k\phi) \ .$$

Indeed, it was this result that motivated our choice of u as the basic variable. Nearly circular orbits can also be nicely treated in terms of u, so this encouraged us to conjecture that all bound orbits can be found analytically to good accuracy.

Our analytic strategy for integrating equation (7) more generally is to replace $S(u)$ with a quadratic function $S_Q(u)$, which has precisely the same zeros, u_a and u_p, corresponding to the apocentre and pericentre of the orbit. Thus $S_Q = q^2(u_p - u)(u - u_a)$ whose q^2, the coefficient of $-u^2$ in S_Q, is to be determined so that in some average sense $S_Q(u)$ is a good approximation to $S(u)$ in the radial range $u_p \geqslant u \geqslant u_a$ occupied by the orbit. For example, we find that choosing q^2 so that $\int_{u_a}^{u_p} S(u)u^{-3/2}du = \int_{u_a}^{u_p} S_Q(u)u^{-3/2}du$ gives a q that is good to 2% accuracy. A better choice, given later, is the natural starting point for the perturbation theory of Appendix A.

Once $S(u)$ in (7) has been replaced by the quadratic

$S_Q(u)$, the integration is easy. From (2), $e = (u_p - u_a)/(u_p + u_a)$, so one sets $u_p = \bar{u}(1+e)$ and it follows that $u_a = \bar{u}(1-e)$ so $S_Q = q^2 \left[e^2 \bar{u}^2 - (u - \bar{u}^2) \right]$. If we make the substitution $u = \bar{u}(1 + e \cos \eta)$, we find that the integration of (7) gives $qk\phi = \eta$, so the orbit is

$$(\ell/r)^k = u/\bar{u} = 1 + e \cos(qk\phi) ,$$

which is of the form (1) with $m = qk$.

Crucial to this method of solving for the orbits is the knowledge of u_p and u_a. A critical step in finding them is to regard the r_p and r_a of an orbit as given, in place of its energy and angular momentum. Those can easily be found if r_p and r_a are given, but solving the other way around is usually difficult. Once the orbit has been found, this approximation allows us to determine the radial action and hence the time from pericentre to a given point on the orbit.

Having outlined our general procedure, we now turn to solving the circular and nearly circular orbits using u, rather than r, as the variable.

2.2   Nearly Circular Orbits

For the circular orbits of angular momentum h, the centrifugal force balances gravity, so $h^2 r_c^{-3} = -d\psi/dr = (2 - k)Ar_c^{k-3}$ and so for them $u = u_c = 2 - k$. Also since $\dot{r}$ is zero for them their energy is ε_c where

$$\varepsilon_c = \tfrac{1}{2}h^2/r_c^2 - Ar_c^{k-2} .$$

Also $S(u_c) = 0$, so in our dimensionless variables, *c.f.* equation (8),

$$E_c = -\tfrac{1}{2}k(2 - k)^{(2-k)/k} . \tag{9}$$

We consider first orbits with energies not much above E_c and we set

$$\Delta = -\left(E - E_c \right)/E_c , \tag{10}$$

then Δ is small for nearly circular orbits and one at the energy of escape.

$$E = E_c \left(1 - \Delta \right) ,$$

where E_c is given by (9). For the nearly circular orbits, we expand the obstreperous u^σ term in (8) about $u = u_c$, omitting terms higher than quadratic in $u - u_c$:

$$u^\sigma = u_c^\sigma \left[1 + \left(\frac{u - u_c}{u_c} \right) \right]^\sigma$$
$$\simeq u_c^\sigma \left[1 + \sigma \left(\frac{u - u_c}{u_c} \right) - \frac{1}{2}\sigma(1 - \sigma)\left(\frac{u - u_c}{u_c} \right)^2 \right] ,$$

so inserting this result into (8),

$$S(u) \simeq ku_c\Delta + 2\Delta\left(k - 1\right)\left(u - u_c\right) - q^2\left(u - u_c\right)^2 ,$$

where again the coefficient of $-u^2$ in S_Q is q^2 and here

$$q^2 = \left[1 + \Delta(k - 1) \right]/k . \tag{11}$$

Completing the square on $(u - \bar{u})$, we have

$$S \simeq q^2 \left[e^2\bar{u}^2 - (u - \bar{u})^2 \right] , \tag{12}$$

where $\bar{u} = u_c + (k - 1)\Delta/q^2$ and

$$e^2 = \Delta q^{-4}\bar{u}^{-2} \left[u_c kq^2 + \Delta(k - 1)^2 \right] . \tag{13}$$

Integrating (7) with S given by (12) by writing $u = \bar{u}\left(1 + e \cos \eta \right)$ yields

$$kq\phi = \eta ; \tag{14}$$

so the orbits take the form (1) with $m = kq$. Notice that as $\Delta \to 0$, $q \to k^{-1/2}$ and $m \to \sqrt{k}$.

Whereas these formulae have been derived by neglecting the $(u - u_c)^3$ and higher terms in the expansion of u^σ, it should be realised that the coefficient of the u^σ term itself vanishes at the energy of escape. Thus, despite this neglect, our formulae are exact, not just for Δ small, but also at $\Delta = 1$. Indeed at $\Delta = 1$ we see that $q^2 = 1, \bar{u} = 1$ and $e = 1$. However, even such partial reassurance should not deceive us into believing that formulae (11) and (13) are good enough at intermediate values of e.

2.3   Analysis of General Orbits

In the non-linear régime, we see from the $e = 1$ orbits that u has its mean at 1 rather than at $2 - k$. It makes little sense to expand u^σ about $u_c = 2 - k$ when Δ is not small. Nevertheless we would like to quadrate S, that is, approximate $S(u)$ by some quadratic function. We adopt a very different procedure in the non-linear case. In place of fixing the energy and the angular momentum of an orbit and then determining its shape and size, we choose, instead, a pericentric distance r_p and an apocentric distance r_a. From these it is simple to find exactly what energy and angular momentum are needed. Equivalently we can fix the values of ℓ and e so then $r_p = \ell/(1 + e)^{1/k}$ and $r_a = \ell/(1 - e)^{1/k}$, as can be seen from (1) with $m\phi$ equal to first 0 and then π.

Since $\dot{r} = 0$ at both r_p and r_a, we have for $\varepsilon < 0$

$$\varepsilon + Ar_p^{k-2} - \tfrac{1}{2}h^2 r_p^{-2} = 0 , \tag{15}$$
$$\varepsilon + Ar_a^{k-2} - \tfrac{1}{2}h^2 r_a^{-2} = 0 , \tag{16}$$

which we may solve for h^2 and ε in terms of r_a and r_p or, alternatively, in terms of ℓ and e:

$$\frac{h^2}{2A} = \frac{r_p^{k-2} - r_a^{k-2}}{r_p^{-2} - r_a^{-2}} = \ell^k \frac{(r_p/\ell)^{k-2} - (r_a/\ell)^{k-2}}{(r_p/\ell)^{-2} - (r_a/\ell)^{-2}}$$
$$= \ell^k \frac{(1 + e)^{(2-k)/k} - (1 - e)^{(2-k)/k}}{(1 + e)^{2/k} - (1 - e)^{2/k}} . \tag{17}$$

Multiplying (15) by r_p^2 and subtracting from it $r_a^2 \times$(16), we deduce

$$\frac{-\varepsilon}{A} = \frac{r_a^k - r_p^k}{r_a^2 - r_p^2} = \ell^{k-2} \frac{(1 - e)^{-1} - (1 + e)^{-1}}{(1 - e)^{-2/k} - (1 + e)^{-2/k}}$$
$$= \ell^{k-2} \frac{2e\left(1 - e^2\right)^{(2-k)/k}}{(1 + e)^{2/k} - (1 - e)^{2/k}} . \tag{18}$$

Another alternative, which is the most useful one in the equivalent problem in quantum mechanics, is to consider h and e as given. Then, eliminating ℓ in (18) in favour of h as found from (17), we obtain

$$\frac{-\varepsilon}{A} = \left(\frac{h^2}{A} \right)^{(2-k)/k} g(e) ; \quad E = -g(e) ,$$
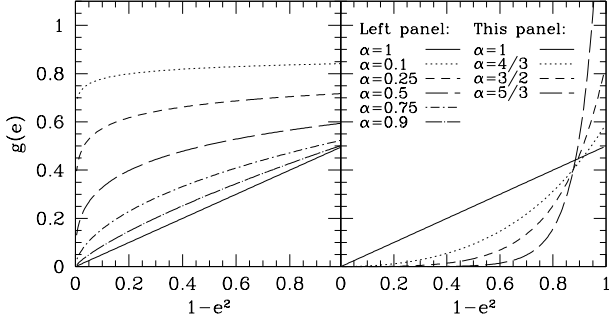
where, setting $\gamma = (2 - k)/k$,

Figure 1. The function g(e) for various values of $\alpha = 2 - k$. g(e) is minus the dimensionless energy E. As $\alpha \to 0$, the graph tends to the line $g(e) \equiv 1$.

$$g(e) = 2^{2/k} e (1 - e^2)^\gamma \frac{[(1+e)^\gamma - (1-e)^\gamma]^\gamma}{[(1+e)^{2/k} - (1-e)^{2/k}]^{2/k}} \ . \qquad (19)$$

Despite its strange appearance, $g(e)$ is not a complicated function. For $k = 1$ it is $\frac{1}{2}(1 - e^2)$ and for $k = 2$ it is 1. We plot g against $1 - e^2$ for several k values in figure 1.

We wish to approximate S by a quadratic in u which must vanish at $u = u_p$ and $u = u_a$, so it has to take the form

$$S \simeq S_Q = q^2 (u - u_a)(u_p - u) = q^2 \left[e^2 \bar{u}^2 - (u - \bar{u})^2 \right] \ , (20)$$

where q is yet to be determined and has been given that notation to conform with our earlier definition that $-q^2$ is the coefficient of u^2 in S. In the above,

$$\bar{u} = \frac{1}{2}(u_a + u_p) = \frac{h^2}{2A}\left(r_p^{-k} + r_a^{-k} \right) = \frac{h^2}{A}\,\ell^{-k} \ , \qquad (21)$$

from which we see that $\bar{u}$ is twice the final expression in (17) but without the ℓ^k. The 'eccentricity' e is $(u_p - u_a)/(u_p + u_a)$ as always. Using (20) for $S(u)$ with the substitution $u = \bar{u}(1 + e\cos\eta)$, we readily integrate equation (7) to obtain $qk\phi = \eta$ as before. So the solution is still equation (1) with $m = qk$, but we must still determine q^2.

A useful approximate formula, good to about 2%, is given by setting $\int_{u_a}^{u_p} S(u) u^{-3/2} du = \int_{u_a}^{u_p} S_Q(u) u^{-3/2} du$. This gives, for $\sigma \neq 1/2$ (i.e. $\alpha \neq 2/3$),

$$q^2 = \frac{\frac{1}{\bar{u}}\left(2 - (\sigma - \frac{1}{2})^{-1}\right) + \left((\sigma - \frac{1}{2})^{-1} - \frac{2}{3}\right)\left(1 + \frac{1}{2}\sqrt{1 - e^2}\right)}{\frac{4}{3}\left(1 - \sqrt{1 - e^2}\right)},$$

$$\qquad (22)$$

with $m = qk$ as before and $\bar{u}$ can be expressed as a function of k and e only, via (17) and (21). We have chosen this power of u in the integrals we equate above, as it gives the best agreement to the true m without compromising on the simplicity of the expression for q. We note that choosing u^{-1} in the integrals gives as good an agreement as using $u^{-3/2}$, but with the resulting m becoming overestimates on the true value for $\alpha < 1$ and vice versa for $\alpha > 1$. Choosing an exponent between -1 and -1.5 results in better agreement still, but we lose the simplicity of the resulting analytic expression for q. The exponent of $-4/3$ is as good as any, but gives the following somewhat awkward result:

$$q^2 = \frac{5(\sigma - 1)(e_+ + e_-)\bar{u}^{-1} + (2 - \sigma)\left[2(e_+ + e_-) + e_+^2 e_-^2\right]}{(3\sigma - 1)(e_+ + e_- - 2e_+^2 e_-^2)},$$

where $e_\pm = (1 \pm e)^{1/3}$.

The angle between successive apocentres is important as such angles accumulate as the orbit is prolonged. We now determine q to get this angle as accurately as possible. It is given by

$$\Phi = \frac{2\pi}{m} = \frac{2}{k}\int_{u_a}^{u_p} S^{-1/2} du = \frac{2}{k}\int_{u_a}^{u_p}\left(\frac{S_Q}{S}\right)^{1/2}\frac{du}{S_Q^{1/2}}$$

$$= \frac{1}{m}\int_{-\pi}^{\pi}\left(\frac{S_Q}{S}\right)^{1/2} d\eta \ ,$$

hence the average of $(S_Q/S)^{1/2}$ over η must be one. We evaluate this average over eight points around $-\pi < \eta < \pi$. There is a difficulty in evaluating $(S_Q/S)^{1/2}$ exactly at the apocentre when $e = 1$ since the apocentre is at infinity. Surprisingly, the result of taking the limit of r_a as $e \to 1$ gives a different (and wrong) result from setting $e = 1$ and then evaluating the limit as $\cos\eta \to -1$. We get around this by using $\cos\eta = -0.990$, where everything is finite, in place of $\eta = \pi$.

At pericentre u_p, both S_Q and S are zero but the limit of $\frac{1}{q}(S_Q/S)^{1/2}$ is

$$\sqrt{\frac{2\bar{u}e}{\sigma(2 - u_p) - 2(u_p - 1)}} \ .$$

We evaluate

$$\frac{1}{q}\sqrt{\frac{S_Q}{S}} = \sqrt{\frac{(u_p - u)(u - u_a)}{2Eu^\sigma + 2u - u^2}} \ ,$$

where $u = \bar{u}(1 + e\cos\eta)$, at the other seven points $\pm\pi/4$, $\pm\pi/2$, $\pm3\pi/4$ and $\cos\eta = -0.990$. For convenience, we label the values of u at $\pm\pi/4$ and $\pm3\pi/4$ as $u = \bar{u}(1 + e/\sqrt{2}) = u_+$ and $u = \bar{u}(1 - e/\sqrt{2}) = u_-$ respectively. Our estimate of $1/q$ is the average over the eight values that result:

$$\frac{1}{q} = \frac{1}{8}\sum_{i=1}^{8}\left[\frac{(u_p - u_i)(u_i - u_a)}{S(u_i)}\right]^{1/2} \ ; \quad m = kq \ . \qquad (23)$$

At each α, the resulting q is a (somewhat complicated) function of e, since E, u_a, u_p, $\bar{u}$ are all functions of e.

2.4   Comparisons with Computed Orbits

Orbits were computed in the x, y plane from the Cartesian form of the equations of motion for potentials with $\alpha = 0.25$, 0.55, 0.75, 1.5 and 1.0. The last provides a valuable check that we get $m = 1$ in the Newtonian case, even at very high eccentricities of order 0.999. We also checked that $m = \sqrt{k}$ for nearly circular orbits and that $m \to k$ as $e \to 1$ for all the values of α. As m varies quite rapidly with eccentricity as $e \to 1$, accurate computations are required at high eccentricities. Orbits in the logarithmic potential which gives a constant circular velocity are considered later, in section 2.6; the approximation adopted there is somewhat different.

Figure 2 shows a comparison between our estimated values of m and the computed values of m for potentials with $\alpha = 0.25$, 0.55, 0.75 and 1.5. For all values of α shown, the deviation of m calculated from the analytical formula (23) is only a fraction of a percent. Notice that both plots are against $\sqrt{1 - e^2}$ so that high eccentricities are on the
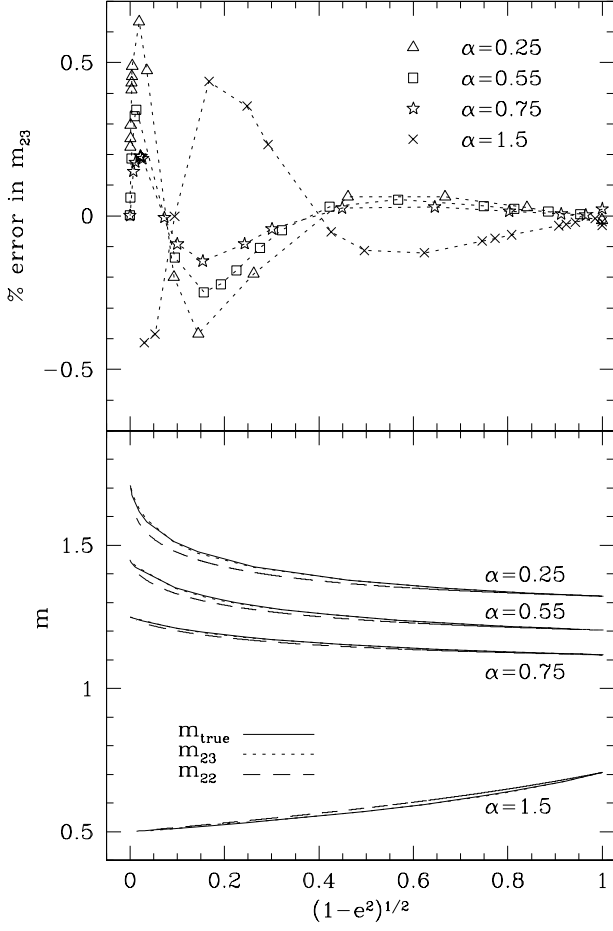
Figure 3. An orbit in the $2 - k = \alpha = 0.25$ potential, chosen to be in the high eccentricity region where the approximation is least good. The full line is the computed true orbit. In panel a), it is compared to $(\ell/r)^k = 1 + e\cos(m\phi)$ (dotted line) with m chosen to agree with the computed orbit. This shows the deviation in the shape of the lobes. In panel b), the value of m is estimated by the 8-point average of equation (23) and the error in m is readily seen from the different precession of the solid and dotted orbits. The difference in the true value of m and that estimated from the 8-point average is less than 0.5%. All orbits start at the starred location.

Figure 4. Two computed orbits with the same $r_{\min}/r_{\max}$ in potentials with different $2-k = \alpha$ are compared with the analytic orbits $(\ell/r)^k = 1 + e\cos(m_{\mathrm{true}}\phi)$ (dotted lines). The $\alpha = 0.55$ orbit has $e = 0.662$ while the $\alpha = 0.75$ orbit has $e = 0.596$ and precesses much less rapidly. Both the shapes and the precession rates of these orbits are well represented by the analytic formula.

Figure 2. The top panel shows the percentage difference between the true value of m from computed orbits and m_{23} calculated from the analytical formula (23), plotted as functions of $\sqrt{1-e^2}$ for different values of $2 - k = \alpha$. The largest deviations occur at high eccentricities; the region 0.1 to 0.3 in $\sqrt{1-e^2}$ corresponds to $0.995 > e > 0.954$. The bottom panel shows the values of m, estimated using (22) and (23), along with the true values, again for four values of α.

left and low eccentricities are on the right. It is, of course, possible to read off m as a function of $\sqrt{1-e^2}$ or of e from the computed points in this figure.

Panel a) of figure 3 shows a computed orbit in the potential with $\alpha = 0.25$ together with an orbit of the same m, r_a and r_p but calculated from the equation $(\ell/r)^k = 1 + e\cos(m\phi)$. This demonstrates how the shape given by equation (1) fits the computed orbit. A better fit is obtained using the perturbation theory of Appendix A. The dotted orbit in panel b) is $(\ell/r)^k = 1 + e\cos(m_{23}\phi)$ and the gradual precession due to the error in the estimated m_{23} is readily seen.

Figure 4 shows two orbits with the same ratio of r_a/r_p but in the potentials with $\alpha = 0.55$ and $\alpha = 0.75$. Because the definition of 'eccentricity' we gave in equation (2) depends on α (through k), these orbits have eccentricities of 0.662 and 0.596 respectively. Notice that the two drawings have the same number of apsides but these have precessed
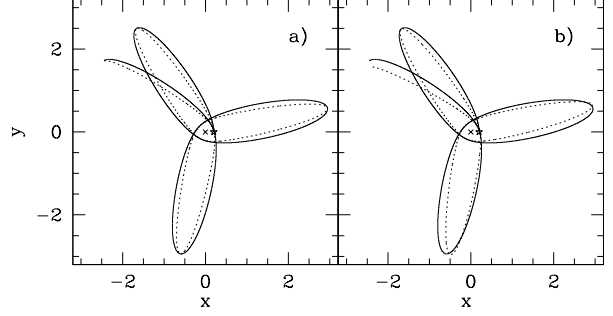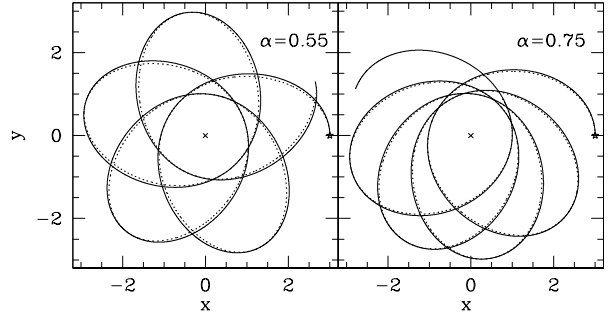
much less for the $\alpha = 0.75$ orbit as the potential is closer to the Keplerian $\alpha = 1$.

Figure 5 shows two orbits in the $\alpha = 1.5$ potential for which the precession is forwards because $\alpha > 1$ whereas the other illustrations all have a backward precession. Under the transformation $\zeta = z^{k/2}$ considered in section 3, these orbits transform into ones in the potentials $\psi \propto r^{2\alpha/k} = r^6$. For orbits with $\alpha > 1$, the straight perturbation theory giving $m = kq$ with q given by (23) yields m to better than 0.5%.

It is often useful to have a vectorial way of delineating orbits and the velocities of particles describing them. To do this, we generalise Hamilton's eccentricity vector which has magnitude e and points toward pericentre. As our pericentres precess within the orbital plane, we invent a rotating eccentricity vector. If we start at pericentre with $\mathbf{e} = \mathbf{e}_0$ we take $\mathbf{e}$ at later times to be given by
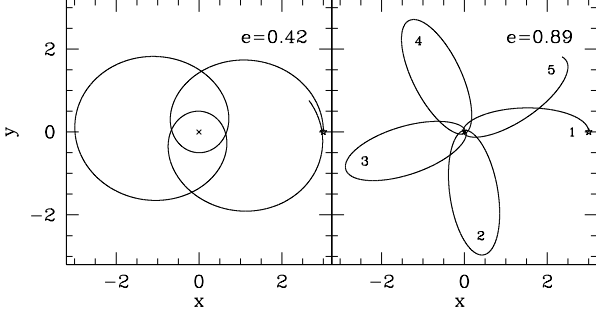
Figure 5. Two orbits in the $2 - k = \alpha = 1.5$ potential. The orbit on the left has a forward precession by nearly $180°$ whilst that on the right has a forward precession by a little over $270°$. The turn close to the origin in the latter cannot be seen but is a more rapidly turning version of that seen on the left. The lobes are numbered in sequential order for the higher eccentricity orbit.

$\mathbf{e} = \mathbf{e}_0 \cos\left[(1 - m)\,\phi\right] + \hat{\mathbf{h}} \times \mathbf{e}_0 \sin\left[(1 - m)\,\phi\right]$. This $\mathbf{e}$ obeys $d\mathbf{e}/d\phi = (1 - m)\,\hat{\mathbf{h}} \times \mathbf{e}$. The angle between the radius vector to the particle and the eccentricity vector is then $m\phi$ and the equation of the orbit (1) can be rewritten

$$(\ell/r)^k = 1 + \mathbf{e}.\hat{\mathbf{r}} \ . \tag{24}$$

The transverse velocity of the particle is clearly $\mathbf{h} \times \hat{\mathbf{r}}/r$ and the radial velocity can be obtained from the orbit and the energy equation. Using our approximations quadrating the latter, we find

$$\mathbf{v} = r^{-1}\left[\mathbf{h} \times \hat{\mathbf{r}} + q\,(r/\ell)^k\,\mathbf{h}.\,(\mathbf{e} \times \hat{\mathbf{r}})\,\hat{\mathbf{r}}\right] \ . \tag{25}$$

     Given $\mathbf{v}$ and $\mathbf{r}$ at one time one might wish to use these equations at a later time. Then one needs to find $\mathbf{e}$, $\mathbf{h}$, ℓ, q and m from the initial $\mathbf{v}$ and $\mathbf{r}$ together with the known potential $\psi = Ar^{-\alpha}$. From $\mathbf{v}$ and $\mathbf{r}$ it is easy to construct $\varepsilon = \frac{1}{2}v^2 - Ar^{-\alpha}$ and $\mathbf{h} = \mathbf{r} \times \mathbf{v}$, from these E is found. For given E and α, e may be found from figure 1. ℓ then follows from (17) and m, q from (23). The direction of $\mathbf{e}$ within the plane perpendicular to $\mathbf{h}$ then follows from $\mathbf{e}.\hat{\mathbf{r}} = (\ell/r)^k - 1$ with the ambiguity in angle resolved from $\mathbf{v}.\hat{\mathbf{r}} = r^{-1}q\,(r/\ell)^k\,\mathbf{h}.\,(\mathbf{e} \times \hat{\mathbf{r}})$ which follows from the $\mathbf{v}$ equation above. Thus all the orbital parameters are determined.

2.5   Action, Adiabatic Invariants and Time

So far we have concentrated on the shape of the orbit in space, however the time from pericentre to any point of the orbit is just as important. Both can be obtained from the action function S_r, whose relationship to $S(u)$ is given below.

$$
\begin{aligned}
S_r = \int p_r dr &= \int \dot{r}\,dr \\
&= \int \sqrt{2\varepsilon + 2Ar^{-\alpha} - h^2 r^{-2}}\ dr \\
&= h \int \sqrt{2\varepsilon h^{-2}r^2 + 2Ah^{-2}r^k - 1}\ r^{-1}dr \\
&= k^{-1}h \int_u^{u_p} \sqrt{S(u)}\ u^{-2}du \ .
\end{aligned}
\tag{26}
$$

If we now use our quadratic approximation we find

$$S_r = qk^{-1}h \int_u^{u_p} \sqrt{e^2 \bar{u}^2 - (u - \bar{u})^2}\ u^{-2}du \ ;$$

setting $u = \bar{u}\,(1 + e\cos\eta)$, this becomes, setting $f = 1/e$,

$$S_r = qk^{-1}h \int_0^\eta \frac{\sin^2\eta}{(f + \cos\eta)^2}\ d\eta \ .$$

Now the related integral

$$
\begin{aligned}
\int_0^\eta \frac{(1 - \cos^2\eta)\,d\eta}{f + \cos\eta} &= \int_0^\eta \left(\frac{1 - f^2}{f + \cos\eta} + f - \cos\eta\right)\,d\eta \\
&= -2\sqrt{f^2 - 1}\,\tan^{-1}\left[\sqrt{\frac{f - 1}{f + 1}}\,\tan\left(\frac{\eta}{2}\right)\right] \\
&\quad + f\eta - \sin\eta \ ,
\end{aligned}
$$

and the integral that we want is just $-d/df$ of this, so

$$
\begin{aligned}
\int_0^\eta \frac{\sin^2\eta}{(f + \cos\eta)^2}d\eta &= -\eta + \frac{2f}{\sqrt{f^2 - 1}}\,\tan^{-1}\left[\sqrt{\frac{f - 1}{f + 1}}\,\tan\frac{\eta}{2}\right] \\
&\quad + \frac{\sin\eta}{f + \cos\eta} \ ,
\end{aligned}
\tag{27}
$$

so, putting this in S_r and remembering that $f = 1/e$,

$$
\begin{aligned}
S_r = qk^{-1}h\Bigg[&-\eta + \frac{2}{\sqrt{1 - e^2}}\,\tan^{-1}\left(\sqrt{\frac{1 - e}{1 + e}}\,\tan\frac{\eta}{2}\right) \\
&+ \frac{e\sin\eta}{1 + e\cos\eta}\Bigg] \ .
\end{aligned}
\tag{28}
$$

The adiabatic invariant is given by

$$J_r = \frac{1}{2\pi}\oint p_r dr = \left(\frac{1}{\sqrt{1 - e^2}} - 1\right)qk^{-1}h \ . \tag{29}$$

Now

$$\partial J_r/\partial\varepsilon|_h = \frac{1}{2\pi}\oint \frac{1}{\dot{r}}dr = P_r/(2\pi) \ ,$$

where P_r is the radial period while

$$-\partial J_r/\partial h|_\varepsilon = \frac{1}{2\pi}\oint hr^{-2}\frac{1}{\dot{r}}dr = \frac{1}{2\pi}\oint \dot{\phi}dt = \Phi/(2\pi) \ .$$

J_r/h is a function of eccentricity, so the partial differentiation is best done via

$$\partial(J_r/h)/\partial\varepsilon = d(J_r/h)/de\ (\partial e/\partial\varepsilon)_h$$

and

$$(\partial J_r/\partial h)_\varepsilon = J_r/h + \frac{d}{de}(J_r/h)\,h\left(\frac{\partial e}{\partial h}\right)_\varepsilon$$

$$(\partial S_r/\partial\varepsilon)_h = \int \frac{dr}{\dot{r}} = t \ , \tag{30}$$

so this expression gives the time to any chosen point in the orbit. In practice, S_r/h is a function of e and η so the partial derivative is done using

$$(\partial S_r/\partial\varepsilon)_h = (dS_r/de)_h\ (\partial e/\partial\varepsilon)_h \ .$$

    In the general case, use of $\frac{1}{u} \propto r^k$ as a variable does not lead to a prettier equation for t, such as the one Kepler derived for $k = 1$, but see the next section for logarithmic potentials.

In the equatorial plane the total action is $S_A = S_r + h_z\phi$. The action variables are J_r and $h_z = h$. The angle variables are the phases of the oscillations in r and ϕ and are given by $w_r = \partial S_A/\partial J_r$ and $w_\phi = \partial S_A/\partial h$. For general orbits, the action variables are most often employed when the potential is of the more general separable form; $\psi = Ar^{-\alpha} - B(\theta)/r^2$; h^2 is no longer conserved but $I = h^2 - 2B(\theta)$ is. The general action is then $S_A = S_r + S_\theta + h_z\phi$ with $S_\theta = \int_{\pi/2}^{\theta} \sqrt{I - h_z^2 \mathrm{cosec}^2\theta}\, d\theta$ and $J_\theta = \frac{1}{2\pi}\oint \partial S_\theta/\partial\theta\, d\theta$. The angle variables are $w_\theta = \partial S_A/\partial J_\theta$ and $w_\phi = \partial S_A/\partial h_z$.

2.6 Logarithmic Potentials $\alpha \to 0$

For small α we write $\psi = Ar^{-\alpha} = Ar_0^{-\alpha}e^{-\alpha\ln(r/r_0)}$ and expand to obtain $\psi = Ar_0^\alpha\left[1 - \alpha\ln(r/r_0) + 0\left(\alpha^2\right)\right]$. We set $A = V^2/\alpha$ and consider taking the limit as $\alpha \to 0$ while keeping V^2 fixed so A tends to infinity. To keep a finite potential, we have to subtract the constant $Ar_0^{-\alpha}$ from ψ, so we obtain a new potential

$$\Psi = \psi - Ar_0^{-\alpha} = -V^2\ln(r/r_0) \ .$$

To apply the methods of section 2.1, we define

$$u = h^2/\left(V^2 r^2\right)$$

and consider orbits defined by pericentric and apocentric distances r_p and r_a. In place of equations (15) and (16) we then have

$$h^2 = V^2\frac{\ln(r_a/r_p)^2}{r_p^{-2} - r_a^{-2}} = \frac{1}{2}V^2 r_p^2(1+e)e^{-1}\ln\left(\frac{1+e}{1-e}\right) \quad (31)$$

so $u_p = (1+e)\bar{u}$, where $\bar{u} = \frac{1}{2e}\ln\left(\frac{1+e}{1-e}\right)$, and $\quad (32)$

$$\varepsilon = V^2\frac{r_a^2\ln(r_a/r_0) - r_p^2\ln(r_p/r_0)}{r_a^2 - r_p^2}$$
$$= \frac{V^2}{4e}\left[(1-e)\ln(1+e) - (1+e)\ln(1-e) + 2e\ln\left(\frac{u_0}{\bar{u}}\right)\right] \ . \quad (33)$$

The orbital equation reads

$$d\phi = \frac{dr}{r\sqrt{2\varepsilon h^{-2}r^2 - 2V^2h^{-2}r^2\ln(r/r_0) - 1}} = \frac{-du}{2\sqrt{S_L(u)}} \ ,$$

where

$$S_L(u) = 2Eu - u\ln(u_0/u) - u^2 \ ,$$
$$= \frac{u}{2e}\left[(1-e)\ln(1+e) - (1+e)\ln(1-e)\right]$$
$$+ u\ln\left(\frac{u}{\bar{u}}\right) - u^2 \quad (34)$$

and $E = \varepsilon/V^2$ which is given in terms of e via (33). We now approximate $S_L(u)$ by the quadratic (20), which shares the same zeros.

As in section 2.3, a useful analytic expression for q is given by again equating

$$\int_{u_a}^{u_p} S_L(u)u^{-3/2}du = 4\bar{u}^{1/2}\left(\sqrt{1+e} - \sqrt{1-e}\right) \times$$
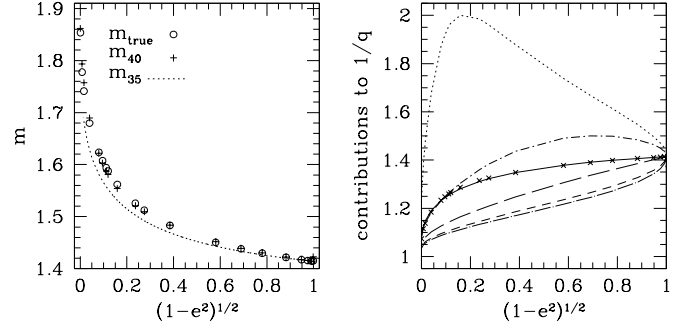$$\left[\frac{\bar{u}}{3}(2 + \sqrt{1-e^2}) - 1\right]$$

Figure 6. The left panel shows a comparison between the true value of m (open circles) and that estimated from equation (35), which gives m to better than 2%, and the 8-point average in equation (40), which gives m to better than 1%. The right panel shows (in thick solid) the true value of $1/q = k/m$ overlaid (in crosses) with that derived from the 8-point average. The agreement is clearly good to a fraction of a percent. The individual contributions made by each of the points given in the terms of equations (39) and (40) are also shown. From the top, the various lines correspond to u_a (dotted), u_+ (dot short-dash), $\bar{u}$ (long dash), u_- (short dash) and u_p (dot long-dash).

to

$$\int_{u_a}^{u_p} S_Q(u)u^{-3/2}du = \frac{8}{3}q^2\bar{u}^{3/2}\left(\sqrt{1+e} - \sqrt{1-e}\right) \times$$
$$\left(1 - \sqrt{1-e^2}\right) \ ,$$

where $\bar{u}$ is given by (32). This yields

$$q^2 = \frac{2 + \sqrt{1-e^2} - 3/\bar{u}}{2\left(1 - \sqrt{1-e^2}\right)} \ ; \qquad m = 2q \ . \quad (35)$$

This expression is good to 2%. Once again, more accurately we calculate q^{-1} from the average of $\sqrt{S_Q/S_L}$ over η, where $u = \bar{u}(1 + e\cos\eta)$. At u_p,

$$2E = \ln(u_0/u_p) + u_p \ ;$$
$$dS_L/du|_p = 2E - \ln(u_0/u_p) + 1 - 2u_p = 1 - u_p \ ;$$
$$dS_Q/du|_p = -2q^2\bar{u}e \ ;$$

so $\sqrt{S_Q/S_L}\Big|_p \to q\sqrt{2\bar{u}e/(u_p - 1)} = q\mu_p \ , \quad (36)$

and at $\eta \pm \pi/2$, $\sqrt{S_Q/S_L} = q\bar{u}e\bar{S}^{-1/2} \ , \quad (37)$

where $\bar{S} = S_L(\bar{u})$. As found in section 2.3, the apocentre once again poses a problem and we evaluate

$$\mu_a = \sqrt{\frac{(u_p - u)(u - u_a)}{S_L(u)}} \quad (38)$$

at $u = \bar{u}(1 + \lambda e)$ where $\lambda = -0.990$, as before, and where $S_L(u)$ is given by (34). So a 4-point estimate of q is given by

$$q^{-1} \simeq q_4^{-1} = \left[\mu_p + \mu_a + 2\bar{u}e\bar{S}^{-1/2}\right]/4 \ , \quad (39)$$

and a 8-point estimate is likewise

$$q^{-1} \simeq q_8^{-1} = \left[4q_4^{-1} + \sqrt{2}\bar{u}e(S_+^{-1/2} + S_-^{-1/2})\right]/8 \ , \quad (40)$$
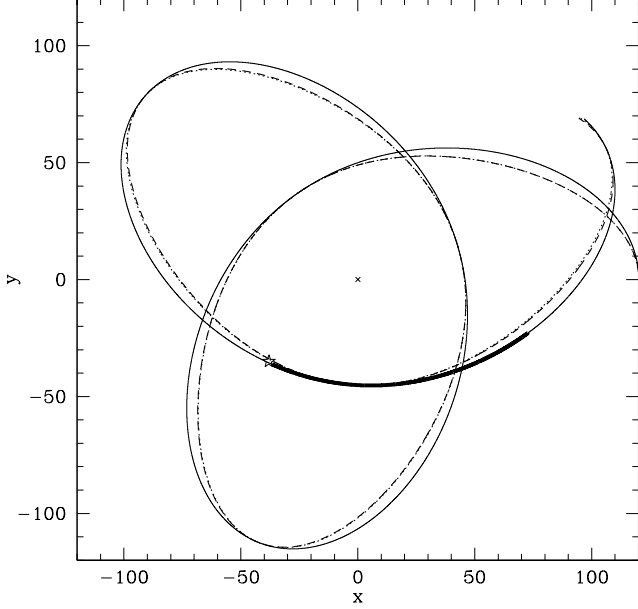
Figure 7. An orbit in the logarithmic potential. The solid line is the computed orbit, and the dotted line shows the approximate orbit $(\ell/r)^2 = 1 + e\cos(m_{\text{true}}\phi)$. Overlaid (and hardly visible) is a dashed line showing an orbit with the estimated m, calculated from (40), instead of m_{true}. The error in this value of m is less than 0.1%. The heavy line shows the piece of such an orbit for the trailing Magellanic Stream, with the location of the Magellanic Clouds indicated by the open star.

where $S_{\pm} = S_L(u_{\pm})$ and $u_{\pm} = \bar{u}(1 \pm e/\sqrt{2})$. Figure 6 shows the contribution of each point in (39) and (40) along with a comparison of the true m and the value derived from the 8-point estimate via $m_{40} = q_8 k$. Figure 7 compares an approximate orbit to a computed one.

The time to a given point in the orbit is given by

$$t = \int_{r_p}^{r} \frac{dr}{\dot{r}} \simeq \frac{h}{2qV^2} \int_{u_p}^{u} \frac{-du}{u\sqrt{(\bar{u}e)^2 - (u - \bar{u})^2}}$$
$$= \frac{h}{2\bar{u}qV^2} \int_0^{m\phi} \frac{d\eta}{1 + e\cos\eta}$$
$$= \frac{h}{\bar{u}qV^2} \frac{1}{\sqrt{1-e^2}} \tan^{-1}\left[\sqrt{\frac{1-e}{1+e}} \tan(q\phi)\right], \quad (41)$$

so the radial period is given by

$$P_r = \frac{\pi h}{\bar{u}qV^2}\left(1 - e^2\right)^{-1/2}. \quad (42)$$

However, a much more interesting result comes from following Kepler, whose equation comes, not from integrating the u equation directly, but by first making the substitution $v = 1/u = V^2 h^{-2} r^2$. This gives

$$t = \frac{h}{2qV^2\bar{u}\sqrt{1-e^2}} \int_{v_p}^{v} \frac{dv}{\sqrt{\bar{v}^2 e^2 - (v - \bar{v}^2)}} = \frac{h}{2qV^2\bar{u}\sqrt{1-e^2}}\chi,$$
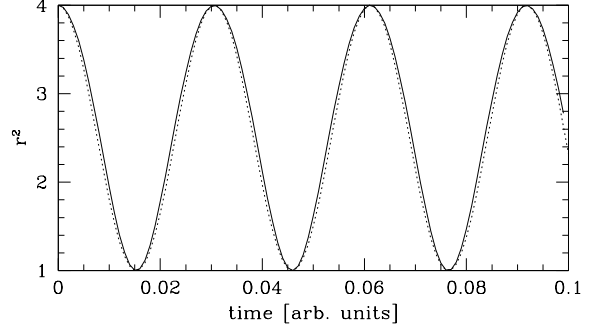

Figure 8. The almost simple harmonic variation of r^2 as a function of time in an orbit with $r_a/r_p = 2$ in the logarithmic potential. The dotted line is a sinusoid of the same period.

where we have written $v = \bar{v}(1 - e\cos\chi)$ and $\bar{v}^{-1} = \bar{u}(1-e^2)$; setting $\kappa = 2qV^2\bar{u}\sqrt{1-e^2}h^{-1}$ we see that

$$\chi = \kappa t \quad \text{and} \quad r^2 \simeq \frac{h^2 V^{-2}}{\bar{u}(1-e^2)}[1 - e\cos(\kappa t)], \quad (43)$$

so with this approximation r^2 vibrates harmonically. Figure 8 shows the computed $r^2(t)$ for an orbit together with the harmonic approximation. The adiabatic invariant is given by $J_r = \frac{1}{2\pi}\oint \dot{r}dr \simeq \frac{h}{4\pi}\oint \sqrt{S(u)}u^{-2}du$. This integral was evaluated in equation (28), so for this case $k = 2$ and

$$J_r = \frac{1}{2}hq\left[\left(1 - e^2\right)^{-\frac{1}{2}} - 1\right]. \quad (44)$$

It should be emphasised that while we have set ourselves the target of getting analytical formulae that give m to 1% or better for all eccentricities, we have not paid attention to minimising errors in the temporal periods. We find that such errors are indeed higher and no doubt our formulae could be improved upon by a study of such errors.

3 TRANSFORMATION THEORY

Newton (1687) realised that the ellipse was a possible orbit both in a harmonic central potential and in an inverse square law. In the first case the centre of force is at the centre of the ellipse, while in the latter case it is at the focus. This led him to pose the question under what circumstances can the same curve be the trajectory of a particle under a force from one of two different centres. Newton's (1714) discussion of this is well described in Chandrasekhar's (1995) book, as is later work by Bohlin (1911), Levi-Civita (1924) and Arnold (1990). All demonstrate the transformation that converts the harmonic ellipse into the Kepler ellipse and vice versa. Collas (1981) gave the relationship between equivalent potentials. Here we show that this transformation, S_1, can be embedded into a larger set of transformations that form a group. We mainly concentrate on the subgroup of switch transformations which have six members but which give just three related potentials r^{-1}, r^2 and r^{-4} in the Kepler case, since r^{-1} is self conjugate under one of the transformations. In the complete group, these potentials are also related to the isochrone (Henon 1959).

The energy equation of a central orbit of angular momentum h in a potential $\psi(r)$ can be written

$$\frac{1}{2}\dot{r}^2 = \psi + \varepsilon - \frac{1}{2}h^2 r^{-2} \ . \tag{45}$$

Now consider the transformation $\tilde{r} = \rho(r)$, $d\tilde{t} = dt/\tau(r)$. Setting $F = \rho'\tau$ we find

$$\frac{1}{2}\left(\frac{d\tilde{r}}{d\tilde{t}}\right)^2 = F^2\left(\frac{1}{2}\dot{r}^2\right) = F^2\psi + F^2\varepsilon - \frac{F^2}{2}h^2 r^{-2} \ . \tag{46}$$

For this to be an orbital equation like (45), the 3 terms on the right must be $\tilde{\psi}(\tilde{r})$, $\tilde{\varepsilon}$ and $-\frac{1}{2}\tilde{h}^2\tilde{r}^{-2}$, but not necessarily in that order.

The transformation S_1 that leads to the Newton-Levi-Civita-Arnold result is found by taking $\tilde{h}^2 = h^2$, identifying the last terms but switching the roles of the other two. Thus for S_1 we set

$$F^2\psi(r) = \tilde{\varepsilon} \ , F^2\varepsilon = \tilde{\psi} \ , F^2 r^{-2} = \tilde{r}^{-2} \ , \tag{47}$$

so we obtain the transformation

$$\tilde{r}^2 = r^2\psi(r)/\tilde{\varepsilon} \ , \tag{48}$$

with $\tilde{\psi}(\tilde{r}) = \varepsilon r^2/\tilde{r}^2 = \varepsilon\tilde{\varepsilon}/\psi$. Applying this to the power-law potential $\psi = Ar^{-\alpha}$ yields $\tilde{r} = r^{1-\alpha/2}\sqrt{A/\tilde{\varepsilon}}$ so $\tilde{\psi}(\tilde{r}) = \tilde{r}^{2\alpha/(2-\alpha)}\left[\varepsilon\,(\tilde{\varepsilon}/A)^{4/(2-\alpha)}\right]$ (note: $\alpha = 1$ gives $\tilde{\psi} \propto \tilde{r}^2$) the quantity in square brackets is, of course, constant. However we could alternatively leave the first term on the right of (46), identifying it with $\tilde{\psi}$ but switching the roles of the other two terms. This leads us to the transformation S_2 in which

$$F^2\psi = \tilde{\psi} \ ; \ F^2\varepsilon = -\frac{1}{2}\tilde{h}^2\tilde{r}^{-2} \ ; \ -\frac{1}{2}F^2 h^2 r^{-2} = \tilde{\varepsilon} \ ,$$

from which we deduce $\psi r^{+2} \propto \tilde{\psi}$ and $\tilde{r} = r^{-1}a^2$, where

$$a^2 = \frac{1}{2}\frac{h\tilde{h}}{\sqrt{\varepsilon\tilde{\varepsilon}}} \ .$$

So S_2 is an inversion accompanied by the change in potential $\tilde{\psi} \propto \tilde{r}^{-2}\psi\left(a^2/\tilde{r}\right) \propto \tilde{r}^{\alpha-2}$ (for power laws). More generally we may ask that $\frac{1}{2}\left(\frac{d\tilde{r}}{d\tilde{t}}\right)^2 = F^2\frac{1}{2}\dot{r}^2$ but that the three terms $\tilde{\psi}$, $\tilde{\varepsilon}$ and $-\frac{1}{2}\tilde{h}^2\tilde{r}^{-2}$ that constitute this quantity are independent linear combinations of $F^2\psi$, $F^2\varepsilon$ and $F^2\left(-\frac{1}{2}h^2 r^{-2}\right)$. If one applies two of this more general class of transformations one after the other, it is simple to see that the net result is a transformation of this class, that the identity transformation belongs to the class and that every transformation has a unique inverse in the class. These transformations form a group in the sense of group theory, since they clearly obey the associative law

$$(T_3 T_2)\,T_1 = T_3\,(T_2 T_1) = T_3 T_2 T_1 \ .$$

Each transformation now gives a new $\tilde{r}(\tilde{t})$ in the new potential $\tilde{\psi}(\tilde{r})$ that corresponds to the old $r(t)$ in the old potential ψ. To get the transformation of ϕ we remember that $\rho' = d\tilde{r}/dr$ and for S_1, $\tilde{h} = h$ so under the transformation S_1 using (46) and (47)

$$d\tilde{\phi} = \frac{\tilde{h}}{\tilde{r}^2}\frac{d\tilde{r}}{d\tilde{r}/d\tilde{t}} = \frac{h}{r}\frac{d\ln\tilde{r}}{\dot{r}} = \frac{d\ln\tilde{r}}{d\ln r}d\phi \ ,$$

but from (48) $\tilde{r} = r\sqrt{\psi/\tilde{\varepsilon}}$ so

$$\frac{d\ln\tilde{r}}{d\ln r} = \left(1 + \frac{1}{2}\frac{d\ln\psi}{d\ln r}\right) \ .$$

This equation takes a particularly simple form when $\tilde{r}$ is a power of r, for then $\tilde{\phi} \propto \phi$. Furthermore, this occurs if and only if ψ follows a power law in r; so

$$\psi = Ar^{-\alpha} \ , \ d\tilde{\phi} = \left(1 - \frac{\alpha}{2}\right)d\phi \ , \ \tilde{r} = r^{1-\frac{\alpha}{2}}\left(\frac{A}{\tilde{\varepsilon}}\right)^{\frac{1}{2}}$$

$$\tilde{\psi} = \varepsilon\left(\frac{\tilde{\varepsilon}}{A}\right)r^\alpha = \varepsilon\left(\frac{\tilde{\varepsilon}}{A}\right)^{\frac{2}{k}}\tilde{r}^{\frac{2\alpha}{k}} \ , \ \tau = \left(1 - \frac{\alpha}{2}\right)^{-1}\left(\frac{\tilde{\psi}}{\varepsilon}\right) \ .$$

Notice that the new potential depends on the energy of the old orbit, so a pair of orbits of different energies in the old potential will map into a pair of orbits in two **different** new potentials that differ by a constant factor, *c.f.* Rosquist & Pucacco (1995). If we write $z = re^{i\phi}$ and $\tilde{z} = \tilde{r}e^{i\tilde{\phi}}$ then, from the above, the S_1 mapping is of the form $\tilde{z} \propto z^{1-\alpha/2}$, i.e. a conformal map in the complex plane. In general, a closed orbit will map into an unclosed Lissajoux rosette, but when $1 - \alpha/2 = N_1/N_2$ where N_1 and N_2 are relatively prime integers, then the transformation of an orbit that closes after one turn will be an orbit that closes after N_2 turns which has N_1 times as many apsides.

For $\alpha = 1$, we have the famous example that transforms Kepler's ellipse into the simple harmonic oscillator. This is

$$\tilde{r} = r^{\frac{1}{2}}(A/\tilde{\varepsilon})^{\frac{1}{2}} \ , \quad \tilde{\phi} = \tfrac{1}{2}\phi \ , \quad \tilde{\psi} = \varepsilon(\tilde{\varepsilon}/A)^2\tilde{r}^2$$

If we apply S_1 again, this time starting with $\tilde{\alpha} = -2$ we find

$$S_1\,[\tilde{r}] \propto \tilde{r}^2 \propto r$$

and

$$S_1\left[\tilde{\psi}\right] \propto r^{-1}$$

so apart from a possible rescaling, the double transformation S_1^2 leads us back to the beginning. This is true generally, not just for power laws, since from equation (48),

$$S_1\,[\tilde{r}] \propto \tilde{r}\sqrt{\tilde{\psi}} \propto r\sqrt{\psi}/\sqrt{\psi} \propto r \quad \text{and} \quad S_1\left[\tilde{\psi}\right] \propto \frac{1}{\tilde{\psi}} \propto \psi \ .$$

We shall ignore the dull rescalings in what follows and write $S_1^2 = I$, the identity. This is in agreement with the concept that a repeated switch leads to no transformation.

We now apply the transformation S_1 to orbits in one of our potentials $\psi = Ar^{-\alpha}$ with $0 < \alpha < 2$. The new potential will be $\tilde{\psi} \propto \tilde{r}^{2\alpha/k}$, which will be a positive power of $\tilde{r}$ and the transformed orbit takes the form

$$\left(\frac{\tilde{\ell}}{\tilde{r}}\right)^2 = 1 + e\cos\left(\frac{2m}{k}\tilde{\phi}\right) \ ,$$

which is indeed an ellipse when $m = k$ as for the Kepler case, which transforms to the harmonic potential. Remarkably, it is always $\tilde{r}^{-2}$ on the left whatever α we start from, but the values of $2m/k$ vary with α.

S_1 is the basis for the regularization of the close encounters of two bodies carried out in three dimensions by Kustaanheimo & Stiefel (1965).

3.1   The Switch Subgroup

If we try to find a transformation that switches the angular momentum and potential terms in (46) while leaving the energy term unchanged, we fail because $\tilde{\varepsilon} = \text{const} = \varepsilon$ and

with F constant we are unable to accomplish the desired switch.

However, we may apply first S_1 and then S_2:

$$S_1\,[r]=r\sqrt{\frac{\psi}{\tilde\varepsilon}} \;\;;\qquad S_1\,[\psi]=\varepsilon\tilde\varepsilon/\psi\;;$$

$$S_2\,[S_1[r]]=\frac{\tilde h\,S_2\big[\tilde h\big]}{2\sqrt{\tilde\varepsilon\,S_2[\tilde\varepsilon]}}\sqrt{\frac{\tilde\varepsilon}{\psi}}\frac1r=\frac{\tilde h\,S_2\big[\tilde h\big]}{2\sqrt{S_2[\tilde\varepsilon]}}\frac{1}{r\sqrt\psi}\;;$$

$$S_2\,[S_1[\psi]]=-\frac{2S_2[\tilde\varepsilon]\,\varepsilon}{\tilde h^2}\,r^2\;.$$

This transformation is not the one we obtain by applying S_2 first and then S_1:

$$\tilde r=S_2\,[r]=\frac12\frac{h\tilde h}{\sqrt{\varepsilon\tilde\varepsilon}}\frac1r\;;\qquad S_2\,[\psi]=-\frac{2\tilde\varepsilon}{h^2}\psi r^2\;;$$

$$S_1\,[S_2[r]]=\frac{\tilde h}{(-2\varepsilon S_1\,[\varepsilon])^{\frac12}}\sqrt\psi\;;$$

$$S_1\,[S_2[\psi]]=\frac{\tilde\varepsilon S_1\,[\tilde\varepsilon]}{\tilde\psi}=-\tfrac12 h^2 S_1\,[\tilde\varepsilon]\,/\left(\psi r^2\right)\;.$$

Applying this double transformation twice gives

$$r_4\propto 1/\left(r\sqrt\psi\right)\text{ and }\psi\propto r^2\;,$$

which is the same as S_2S_1 up to constants of proportionality, while a further application of S_1S_2 gives

$$r_6\propto r\qquad\text{and}\qquad \psi_6\propto\psi\;,$$

so the triple application of S_1S_2 gives a multiple of the identity.

Before going any further, let's see where we can get if we start with $\psi\propto 1/r$. We can get to $\psi\propto r^2$ and back using S_1, but using S_2 leaves $\tilde\psi\propto 1/\tilde r$, so Newton's law is invariant under S_2. However, S_2 acting on r^2 leads to $\tilde\psi\propto\tilde r^{-4}$, but a further application of S_1 leaves r^{-4} invariant. Thus under the transformations considered so far, there are conjugate orbits in the r^{-1}, r^2 and r^{-4} potentials. More generally, if we start with $\psi\propto r^{-\alpha}$, then S_1 gets us to $\psi\propto r^{2\alpha/(2-\alpha)}$ (where we have dropped the tildes), while S_2 brings us to $\psi\propto r^{\alpha-2}$. The double transformations S_2S_1 and S_1S_2 give $\psi\propto r^{-4/(2-\alpha)}$ and $r^{2(2-\alpha)/\alpha}$ respectively, while $S_1S_2S_1$ leads to $\psi\propto r^{-4/\alpha}$. Further applications only bring us back to potentials already included. In fact, there are six transformations in this subgroup, yielding a conjugacy of orbits in the six potentials $r^{-\alpha}$, $r^{2\alpha/(2-\alpha)}$, $r^{\alpha-2}$, $r^{-4/(2-\alpha)}$, $r^{2(2-\alpha)/\alpha}$ and $r^{-4/\alpha}$. For $\alpha=-1$, these are r, $r^{-2/3}$, r^{-3}, $r^{-4/3}$, r^{-6} and r^4. For $\alpha=1/2$, they are $r^{-1/2}$, $r^{2/3}$, $r^{-3/2}$, $r^{-8/3}$, r^6 and r^{-8}. These powers become somewhat bizarre for small α.

$\alpha=1/6$ gives $r^{-1/6}$, $r^{2/11}$, $r^{-11/6}$, $r^{-24/11}$, r^{22} and r^{-24} ;

$\alpha=-1/6$ gives $r^{1/6}$, $r^{-2/13}$, $r^{-13/6}$, $r^{-24/13}$, r^{-26} and r^{24};

degeneracies similar to those for the Kepler potential occur for $\alpha=1$, 4 or ±2.

The simple relationship $\tilde\phi=\phi(1-\alpha/2)$ holds only for power law potentials under the S_1 transformation. Under S_2 we find

$$d\tilde\phi=\tilde h\tilde r^2\left(d\tilde r/d\tilde t\right)^{-1}d\tilde r=-(-2\varepsilon)^{1/2}\,r^{-1}dr/\dot r$$
$$=-\frac{(-2\varepsilon)^{1/2}}{h}rd\phi\;,$$

which no longer gives a simple relationship of $\tilde\phi$ to ϕ. However, $\frac{(-\tilde\varepsilon)^{1/4}}{\tilde h^{1/2}}\int\tilde r^{1/2}d\tilde\phi=\tilde\chi(\tilde r)=-\frac{(-\varepsilon)^{1/4}}{h^{1/2}}\int r^{1/2}d\phi=-\chi$, so if $\chi(r)$ is known for the first orbit then, with $\tilde r(r)$ known, $\tilde\chi(\tilde r)$ is known for the second.

These transformations are not restricted to power-law potentials. Under S_1, Plummer's potential $\psi=\mu\left(r^2+b^2\right)^{-1/2}$ transforms into

$$\tilde\psi=\frac{2\varepsilon b^2\tilde r^{-2}}{1\pm\sqrt{1-\dfrac{4\mu^2b^2}{\tilde\varepsilon^2\tilde r^4}}}\;,\text{ while under }S_2\text{ it becomes}$$

$$\tilde\psi\propto\frac{r^2}{(r^2+b^2)^{1/2}}\propto\frac{1}{\tilde r\left(\dfrac{4\varepsilon\tilde\varepsilon b^2}{h^2\tilde h^2}\tilde r^2+1\right)^{1/2}}\;.$$

It would be tedious to give the complete set but there are six; ψ, $S_1[\psi]$, $S_2[\psi]$, $S_2[S_1[\psi]]$, $S_1[S_2[\psi]]$ and $S_1[S_2[S_1[\psi]]]$.

3.2 The Larger Group

When we ask that $F^2\dot r^2=\left(d\tilde r/d\tilde t\right)^2$ but in place of merely switching the terms on the right of (46) we ask that those terms are linear combinations of $\tilde\psi\tilde\varepsilon$ and $-\frac12\tilde h^2\tilde r^{-2}$, we obtain the full group of transformations. A general transformation of the group is then

$$\tilde\psi=F^2\left(a_{11}\psi+a_{12}\varepsilon-a_{13}\tfrac12 h^2 r^{-2}\right)\;,$$
$$\tilde\varepsilon=F^2\left(a_{21}\psi+a_{22}\varepsilon-a_{23}\tfrac12 h^2 r^{-2}\right)\;,$$
$$-\tfrac12\tilde h^2\tilde r^{-2}=F^2\left(a_{31}\psi+a_{32}\varepsilon-a_{33}\tfrac12 h^2 r^{-2}\right)\;,$$

where $\sum_n a_{nj}=1$ for $j=1,2,3$.

If we take the particular transformation with $a_{31}=a_{32}=a_{13}=a_{23}=a_{12}=0$ then $a_{21}=1-a_{11}$ and $a_{22}=a_{33}=1$ so we get, taking $\tilde h=h$ without loss of generality,

$$\tilde\psi=F^2a_{11}\psi\;,\tilde\varepsilon=F^2\left[(1-a_{11})\psi+\varepsilon\right]\;,\tilde r^{-2}=F^2r^{-2}\;.$$

The second of these gives the relationship of $\tilde r$ to r when the value of F^2 is taken from the third:

$$\tilde r^2=r^2\left[(1-a_{11})\,\psi+\varepsilon\right]/\tilde\varepsilon\;.$$

Taking $\psi=GM/r$ as our initial potential, we readily solve to find $r(\tilde r)=\left(\frac{\tilde\varepsilon}{\varepsilon}\right)^{1/2}\left(\sqrt{\tilde r^2+b^2}-b\right)$,

$$\text{where}\quad b=\frac{(1-a_{11})\,GM}{2\sqrt{\varepsilon\tilde\varepsilon}}\;.$$

The potential $\tilde\psi=\dfrac{a_{11}GMr}{\tilde r^2}=a_{11}\left(\dfrac{\tilde\varepsilon}{\varepsilon}\right)^{1/2}\dfrac{GM}{\sqrt{\tilde r^2+b^2}+b}$,

where we used $\tilde r^2=\left(\sqrt{\tilde r^2+b^2}-b\right)\left(\sqrt{\tilde r^2+b^2}+b\right)$.

The potential $\tilde\psi$ is the isochrone, see Henon (1959). This is the most general potential in which all orbits can be found using only elementary functions (trigonometric etc.) as stated by Eggen, Lynden-Bell & Sandage (1962); the detailed proof of this was only published many years later in Evans et al. (1990).

4 CONCLUSIONS

We have found crude, but useful, approximations to Abelian functions by quadrating the expression under the surd while

keeping the end points as the constant parameters. For our problem, these methods give accuracies to better than 1%.

We have shown that the $e = 1$ 'parabolic' orbits at the energy of escape can be solved exactly, and we have given analytic expressions for $m(e)$ which hold for all eccentricities $0 \leqslant e \leqslant 1$. We have thus illuminated why Struck found these orbits to be such good approximations at low and moderate eccentricities.

The transformation theory has allowed us to extend these results to orbits in potentials which are positive powers of r and we have extended the transformations to form a group.

5 ACKNOWLEDGEMENTS

APPENDIX A: PERTURBATION THEORY

We have $S(u) = 2Eu^\sigma + 2u - u^2$; $\sigma = 2(1 - \alpha)/(2 - \alpha)$, $0 < \alpha < 2$. We rewrite it in the forms;

$$
\begin{aligned}
S(u) &= q^2 \left[(\bar{u}e)^2 - (u - \bar{u})^2 \right] / [1 + p(u)]^2 \\
&= q^2 (u_p - u)(u - u_a) / [1 + p(u)]^2 \ ,
\end{aligned} \tag{A1}
$$

where $p(u)$ is the perturbation function that allows for the difference between $S(u)$ and our quadratic approximation to it. The orbit is given by

$$
qk \, d\phi = q \int \frac{-du}{\sqrt{S(u)}} = \int \frac{(1 + p(u))\, du}{\sqrt{\bar{u}^2 e^2 - (u - \bar{u})^2}} \tag{A2}
$$

$$
= \int \left(1 + p\left[\bar{u}\left(1 + e\cos\eta \right) \right] \right) d\eta \ , \tag{A3}
$$

where $u = \bar{u}(1 + e\cos\eta)$.

Now p may be expanded in a Fourier series in η:

$$
p = a_0 + a_1 \cos\eta + a_2 \cos 2\eta + a_3 \cos 3\eta + a_4 \cos 4\eta + \dots \tag{A4}
$$

where $a_n = \frac{1}{\pi} \int_0^{2\pi} \cos(n\eta)\, d\eta$ for $n \neq 0$. We chose q so that the average of $\sqrt{S_Q/S}$ over η is 1. This ensures that the average $< p >= 0$, so $a_0 = 0$.

Keeping just those terms with $n \leqslant 4$ we find that $p(\eta)$ is given by

$$
p(0) = a_1 + a_2 + a_3 + a_4 \tag{A5}
$$

$$
p(\pi) = -a_1 + a_2 - a_3 + a_4 \tag{A6}
$$

$$
p(\pi/2) = -a_2 + a_4 \tag{A7}
$$

$$
p(\pi/4) = a_1/\sqrt{2} - a_3/\sqrt{2} - a_4 \tag{A8}
$$

$$
p(3\pi/4) = -a_1/\sqrt{2} + a_3/\sqrt{2} - a_4 \ . \tag{A9}
$$

From these, we may deduce the following:

$$
a_1 = \tfrac{1}{4} \left[p(0) - p(\pi) + \sqrt{2}\left(p(\pi/4) - p(3\pi/4) \right) \right] \ ; \tag{A10}
$$

$$
a_2 = \tfrac{1}{4} \left[p(0) + p(\pi) - 2p(\pi/2) \right] \ ; \tag{A11}
$$

$$
a_3 = \tfrac{1}{4} \left[p(0) - p(\pi) - \sqrt{2}\left(p(\pi/4) - p(3\pi/4) \right) \right] \ ; \tag{A12}
$$

$$
a_4 = \tfrac{1}{4} \left[p(0) + p(\pi) + 2p(\pi/4) \right] \ . \tag{A13}
$$

The perturbed orbit is given by the implicit equations

$$
m\phi = \eta + \sum_{n=1}^{4} n^{-1} a_n \sin(n\eta) \ \text{and} \ u = \bar{u}(1 + e\cos\eta) \ . \tag{A14}
$$

In the above solution for the a_n, we have not used the combination $\frac{1}{2}\left[p(\pi/4) + p(3\pi/4) \right]$ because the condition $< p >= 0$ ensures its consistency and we replace $p(\pi)$ by $p(\phi)$ with $\phi = \cos^{-1}(-0.990)$.

APPENDIX B: UNBOUND ORBITS

When $\varepsilon > 0$ the E term dominates at large distances (u small). Indeed when $E > 1$ the potential term never dominates. We write the equation for the orbit in the alternative forms,

$$
\begin{aligned}
k \, d\phi &= -\frac{du}{\sqrt{2Eu^\sigma + 2u - u^2}} = -\frac{du}{\sqrt{E^{k/\alpha} U^{2(1-\alpha)} \Sigma(U)}} \\
&= -\frac{kE^{k/(2\alpha)} dU}{\sqrt{\Sigma(U)}} \ ,
\end{aligned} \tag{B1}
$$

where $\quad \Sigma(U) = 2 + 2U^\alpha - E^{k/\alpha} U^2$

and $\quad U = E^{-1/\alpha} u^{1/k} = E^{-1/\alpha} \left(h^2/A \right)^{1/k} r^{-1}$.

When $E > 1$ we use the U form everywhere and approximate the U^α term in Σ as a quadratic. We specify our orbit by the values of the impact parameter $b = h/\sqrt{2\varepsilon}$ and the value of the perihelion distance r_p. The energy equation at r_p is

$$
\varepsilon = -Ar_p^{-\alpha} + \tfrac{1}{2} h^2 r_p^{-2}
$$

hence $\quad 1 + Ar_p^{-2}\varepsilon^{-1} = b^2/r_p^2 = \beta^2$, $\qquad$ say;

b and r_p are both specified and A is known, so we find ε as

$$
\varepsilon = Ar_p^{-\alpha} / \left(\beta^2 - 1 \right) \ ,
$$

and with ε now known h is given by $h = b\sqrt{2\varepsilon}$.

From (6) we can now deduce the dimensionless energy

$$
E = \left(\frac{\varepsilon}{A} \right) \left(\frac{h^2}{A} \right)^{\alpha/k} = \left[\left(\sqrt{2}\beta \right)^\alpha / \left(\beta^2 - 1 \right) \right]^{2/k} \ .
$$

We may also express U in dimensionless combinations

$$
U = (E)^{-1/\alpha} \left[2\beta^2 / \left(\beta^2 - 1 \right) \right]^{1/k} (r_p/r) \ .
$$

We require our quadratic approximation to Σ to be exact at r_p, that is, $U = U_p$ and exact at $U = 0$ and at the centre of the range $U_p/2$. Then the approximation takes the form

$$
\Sigma \equiv 2 + 2U^\alpha - E^{k/\alpha} U^2 \simeq \left(2/U_p + Q^2 U \right) (U_p - U) \tag{B2}
$$

$$
= Q^2 \left[e^2 \bar{U}^2 - \left(U - \bar{U} \right)^2 \right] \ , \tag{B3}
$$

where

$$
Q^2 = (2/U_p)^2 + 2(2/U_p)^k - E^{k/\alpha} \ ; \quad \bar{U} = \tfrac{1}{2} U_p - \left(Q^2 U_p \right)^{-1} \ ;
$$

$$
e^2 = 1 + 2\left(Q^2 \bar{U} \right)^{-1} \ .
$$

Thus for $E > 1$ we may integrate, using this quadratic approximation to obtain $\frac{u}{\bar{u}} = \frac{\ell}{r} = 1 + e_* \cos m_* \phi$, where $m_*^2 = Q^2 E^{-k/\alpha}$ and $\ell = E^{-1/2} \left[2\beta^2 / \left(\beta^2 - 1 \right) \right]^{1/k} \left(r_p / \bar{U} \right) = \frac{2}{r_p^{-1} + r_a^{-1}}$.

When $0 < E < 1$, the E term dominates at large r (small u) but the $2u$ term dominates it when $1 < U$. We approximate the smaller of these terms in each region and make sure that the two approximations to $S(u)$ join smoothly with the same gradient at $U = 1, u = E^{k/\alpha}$. The pericentre lies in the $U > 1$ region where the $S(u)$ form is appropriate, so we need $S(u_p) = 0$.

$$S(u) \simeq (c_0 + c_1 u)(u_p - u) = c_1 \left[e^2 \bar{u}^2 - (u - \bar{u})^2 \right]; \ u \geqslant E^{k/\alpha},$$

where c_0 and c_1 are constants to be determined and $e, \bar{u}$ follow them.

Since U_p no longer lies in the zone where the Σ form is used, our former approximate form (B2) for Σ is not appropriate. We write, instead

$$\Sigma(U) = 2 + C_1 U - C_2 U^2 = C_2 \left[e_*^2 \bar{U} - \left(U - \bar{U} \right)^2 \right] ,$$

where C_1 and C_2 are constants.

Now looking at (B1), it is $S(u)$ and $E^{k/\alpha} U^{2(1-\alpha)} \Sigma (U)$ that have to be continuous with a continuous derivative at the junction point $u = E^{k/\alpha}$, where we demand the derivatives be exact. We have $du/dU = k E^{k/\alpha}$ there. Continuity requires:

$$\left(c_0 + c_1 E^{k/\alpha} \right) \left(u_p - E^{k/\alpha} \right) = E^{k/\alpha} \left(2 + C_1 - C_2 \right) , \quad \text{(B4)}$$

whilst the condition on the derivative provides:

$$\left(\sigma + 1 - E^{k/\alpha} \right) = c_1 \left(U_p - 2E^{k/\alpha} \right) - c_0$$
$$= k^{-1} \left[4(1 - \alpha) + (3 - 2\alpha)C_1 \right] - 2C_2 . \quad \text{(B5)}$$

Finally, we demand that the value of the approximation be exact at $u_p/2$. This last condition takes a different form dependent on whether $u_p/2$ is greater than or less than $E^{k/\alpha}$ as different forms of approximation hold in those two regions. Thus

$$2 + 2E^{-1} \left(\tfrac{u_p}{2} \right)^{\alpha/k} - E^{-1} \left(\tfrac{u_p}{2} \right)^{2/k}$$
$$= 2 + C_1 E^{-1/\alpha} \left(\tfrac{u_p}{2} \right)^{1/k} - C_2 E^{-2/\alpha} \left(u_p/2 \right)^{2/k}$$
$$\text{for } \tfrac{u_p}{2} < E^{k/\alpha} \text{, and}$$
$$2^{\alpha/k} E + u_p - \tfrac{1}{4} u_p^2 = \left(c_0 + c_1 \tfrac{u_p}{2} \right) \tfrac{u_p}{2} \text{ for } \tfrac{u_p}{2} \geqslant E^{k/\alpha} . \quad \text{(B6)}$$

The four equations (B4), (B5 i), (B5 ii) and (B6), in whichever form is relevant, are readily solved for the four constants c_0, c_1, C_1 and C_2.

Our orbits can now be found in the form, $(\ell/r)^k = 1 + e \cos(m\phi)$ for $h^2/Ar^k \geqslant E^{k/2}$ where $m^2 = k^2 c_1$, but for larger r, we get $\ell_*/r = 1 + e_* \cos \left[m_* \left(\phi + \phi_* \right) \right]$.

REFERENCES

Arnold V. I., 1990, Huygens & Barrow, Newton & Hooke, Birkhäuser Verlag, pp 95–100

Binney J., Tremaine S., 1987, Galactic dynamics. Princeton, NJ, Princeton University Press, 1987

Boccaletti D., Pucacco G., 1996, Theory of Orbits. Volume 1: Integrable Systems and Non-perturbative Methods, XIII, 392 pp. 71 figs.. Springer-Verlag Berlin Heidelberg New York.

Bohlin K., 1911, Bulletin Astronomique, Serie I, 28, 113

Chandrasekhar S., 1995, Newton's Principia for the Common Reader. Oxford University Press, pp 79–125

Collas P., 1981, J. Math. Phys, 22, 2512

Contopoulos G., 2002, Order and chaos in dynamical astronomy. Springer, New York

Eggen O. J., Lynden-Bell D., Sandage A. R., 1962, ApJ, 136, 748

Evans N. W., de Zeeuw P. T., Lynden-Bell D., 1990, MNRAS, 244, 111

Henon M., 1959, Annales d'Astrophysique, 22, 126

Kalnajs A. J., 1979, AJ, 84, 1697

Kustaanheimo P., Stiefel E., 1965, J.reine angew Math, 218, 204

Levi-Civita T., 1924, Questioni di Meccanica Classica & Relativistica. Zanichelli, Bologna

Lynden-Bell D., Lynden-Bell R. M., 1995, Notes and Records of the Royal Society, 51, 197

Newton I., 1687, Principia. Royal Society, London

Newton I., 1714, Principia 2nd Edition. Royal Society, London

Rosquist K., Pucacco G., 1995, J. Phys. A: Math. Gen., 28, 3235

Struck C., 2006, AJ, 131, 1347

Touma J., Tremaine S., 1997, MNRAS, 292, 905

Valluri S. R., Yu P., Smith G. E., Wiegert P. A., 2005, MNRAS, 358, 1273